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# A NEW $\theta$-SCHEME ALGORITHM AND INCOMPRESSIBLE FEM FOR VISCOELASTIC FLUID FLOWS (*)

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Abstract. — *This paper presents a new mixed finite element method for the computation of incompressible viscoelastic fluids flows. The decoupled computation of stresses and velocities is performed with an algorithm which involves a time approximation by alternating direction implicit algorithms. The method is of order two in time and allows fast calculation of stationary solutions. As finite elements, we have used the zero divergence Raviart-Thomas element for approximating the velocities, and the Lesaint-Raviart element for the stresses. Application of the Oldroyd-B fluid in an abrupt contraction is given. The numerical results show that no upper limit of the Weissenberg number is encountered.*

Résumé. — *Nous présentons une nouvelle formulation mixte pour le calcul d'écoulements de fluides viscoélastiques incompressibles. L'approximation en temps du problème est effectuée à l'aide d'une méthode de directions alternées. Ceci nous conduit à un algorithme permettant de découpler le calcul des vitesses de celui des contraintes. D'ordre deux en temps, cette méthode permet de plus le calcul rapide de solutions stationnaires. L'élément à divergence nulle de Raviart-Thomas est utilisé pour les vitesses, et celui de Lesaint-Raviart pour les contraintes. La méthode est appliquée au problème de l'écoulement d'un fluide d'Oldroyd dans une contraction brusque. Les résultats numériques ont été effectués sans rencontrer de nombre de Weissenberg limite.*

## 1. INTRODUCTION

The spectacular effects occurring in viscoelastic fluid flows have been extensively described in many books and papers (for example [1, 5]) and cannot be predicted by Navier-Stokes equations. These phenomena are

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mainly related to memory and elongational effects of the material, which can be represented by a suitable rheological model. In this context, numerical simulation may be considered as an important tool for prediction of phenomena, as vortex flows which are of interest in polymer processing. In the isothermal case, the basic equations of the problem are mass and momentum equations to be solved together with a rheological constitutive equation. The relevant set of equations is generally non-linear.

In relation to numerous applications and fundamental problems arising in the field of non-Newtonian flows, intensive research has revealed the main problems and theoretical difficulties for solving the large systems obtained from the discretisation of the non-linear boundary value problems. We may refer to Keunigs's book [16] as an exhaustive historical document on the considerable work performed in viscoelastic flow calculations. We now briefly recall and comment the main features of previous work reported before 1987, and the more recent theoretical and numerical results beyond this date. As many authors, we still find that it is convenient to define two simple non-dimensional numbers $Re$ and $We$ to characterize the viscoelastic fluid flows. The Reynolds number $Re$ is given by

$$Re = \rho \frac{UL}{\eta_n} \tag{1.1}$$

where $\eta_n$ is the (constant) viscosity, $\rho$ is the density, $U$ is the characteristic velocity and $L$ is a characteristic length. The elastic number, $We$, generally called the Weissenberg number (or Deborah number) is expressed by :

$$We = \frac{\lambda U}{L} \tag{1.2}$$

where $\lambda$ is an average characteristic time of the fluid.

### 1.1. Main results before 1987

The numerical results have indicated a divergence of the algorithms beyond $We = 4$. At that time, research has more investigated the problem of computational « unstabilities » than realistic physical features. Although the failure of the algorithms can be related to the theoretical foundations of the constitutive models, we may generally question on flow problems occurring close to singularities of complex geometries and theoretical problems due to prescription of boundary conditions. Those numerical methods also lead losses of convergence, which may explained as follows :

(i) non-compatibility between the approximating spaces for the stresses and the velocities ;

(ii) the non-consideration of the hyperbolicity [15] of equations in relation

to boundary conditions to be involved, and the approximating schemes for the derivatives leading to wiggles for the solutions ;

(iii) the non-efficiency of the algorithms : the computing times are enormous and even macro-computers might be saturated.

### 1.2. In 1987 : Marchal and Crochet

In 1987, Marchal and Crochet [4] have presented the streamline upwind technique introduced by Hughes and Brooks [3] for convection problems. Using particular elements for approximating the velocities and stresses, numerical results were obtained for high $We$. The authors have adopted a continuous finite element approximation for the stresses.

To obtain a *well-posed* problem [9], as well as acceptable solutions in the Stokes problem (at $We = 0$), these authors used meshes involving sub-elements.

Then, the resulting high order matrices arising from the discretized system of equations had to be solved by use of a CRAY-XMP, with an important computing time.

### 1.3. The recent work of M. Fortin and A. Fortin

The recent work of M. Fortin and A. Fortin [7] is also to be underlined. By developing a non-continuous finite element approximation for the stresses, the authors have proved that the problem is well-posed for $We = 0$ and have used a discontinuous method proposed by Lesaint and Raviart [17].

However, when using high-order finite elements, the corresponding scheme does not verify the TVD (Total Variation Decreasing) condition [12], and oscillating approximate solutions are obtained. The method still involves large degrees of freedom, and the decoupled technique leads to high computing times (on work stations), which prevents from further numerical experiments.

### 1.4. The present work

In the present work, we develop a finite element method applied to the Oldroyd-B constitutive model. The basic equations and the related mathematical problems are presented in section 2. Taking into account the incompressibility condition for the fluid (which implies the zero divergence for the velocity vector), we consider the incompressible finite elements of Raviart and Thomas [10, 23].

In section 3, some original approximations of the stresses are proposed, and are shown to correspond to a well-posed problem at $We = 0$. In particular, for elements of lower degree, a minimum size of the discretized

problem is obtained (asymptotically, one degree-of-freedom by scalar function and by element).

In section 4, on the basis of the work of Glowinski and Périaux [11] on Navier-Stokes equations, we present a new algorithm using the alternating directions implicit method. This enables us to decouple the difficulties involved by the non-linearity of the equations. The non-stationary approach is used to obtain solutions in steady flow situations.

## 2. FORMULATION OF THE PROBLEM

In this section, we consider the general laws of conservation of the incompressible isothermal flow and the rheological constitutive equation of the fluid. The boundary condition equations to be prescribed are also given.

### 2.1. The Oldroyd-B constitutive equation

The Cauchy stress tensor may be expressed as :

$$\sigma = -pI + 2\,\eta_n D(\mathbf{u}) + \tau \tag{2.1}$$

where $p$ is the hydrostatic pressure, $\mathbf{u}$ is the velocity vector, $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$ the rate-of-deformation tensor, $\tau$ the extra-stress and $\eta_n > 0$ denotes the « solvent viscosity ». For the extra-stress tensor, we consider the differential Oldroyd-B model [1, 19] of equation :

$$\lambda \overset{\square}{\tau} + \tau = 2\,\eta_v D(\mathbf{u}) \tag{2.2}$$

where $\lambda > 0$ is the relaxation time, $\eta_v > 0$ is the « elastic » viscosity. The symbol $(^{\square})$ is related to the objective derivation of a symmetric tensor [12, 2] :

$$\overset{\square}{\tau} = \frac{\partial \tau}{\partial t} + (\mathbf{u}\,.\,\nabla)\,\tau + \tau W(\mathbf{u}) - W(\mathbf{u})\,\tau - a(D(\mathbf{u})\,\tau + \tau D(\mathbf{u})) \tag{2.3}$$

with $a \in [-1, 1]$ and $W(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^t)$ denote the vorticity tensor.

### 2.2. Conservation laws

The mass conservation equation may be written as :

$$\operatorname{div} \mathbf{u} = 0 \tag{2.4}$$

for a fluid of constant specific mass $\rho > 0$.

The momentum equation is :

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \,.\, \nabla)\, \mathbf{u} \right) - \mathbf{div}\ \tau - \eta_n \Delta \mathbf{u} + \nabla p = 0 \,. \tag{2.5}$$

In the following, we investigate the case of slow flows. Thus, the inertia term $(\mathbf{u} \,.\, \nabla)\, \mathbf{u}$ may be ignored.

The conservation of moments leads to symmetry properties of the tensor $\sigma$ : $\sigma = \sigma^t$. Then, from (2.1) :

$$\tau = \tau^t \,. \tag{2.6}$$

### 2.3. Boundary conditions

The conservation laws (2.4)-(2.5) and the Oldroyd constitutive equation (2.2) have to be used together with initial and boundary conditions. For $a = 1$ in (2.3) and $\eta_n > 0$, the set of equations (2.2), (2.4)-(2.5) is found to be of mixed parabolic-hyperbolic type [14, 15]. The characteristic lines are the streamlines, and the components of the stress tensor $\tau$ may be considered as quantities conveyed on these characteristics.

Let $\Omega$ be a finite connected flow domain of $\mathbf{R}^2$ or $\mathbf{R}^3$. The required boundary and initial conditions are the following :

(i) a condition of Dirichlet type for the velocities, on the boundary $\Gamma = \partial\Omega$ :

$$\mathbf{u} = \mathbf{u}_\Gamma \quad \text{on} \quad \Gamma \tag{2.7}$$

(ii) a condition for the stresses on the upstream boundary section

$\Gamma_- = \{x \in \Gamma \,;\, (\mathbf{u} \,.\, \mathbf{n})(x) < 0\}$ :

$$\tau = \tau_\Gamma \quad \text{on} \quad \Gamma_- \tag{2.8}$$

where $\mathbf{n}$ is the outward unit normal vector to $\Omega$ at the boundary $\Gamma$.

(iii) At time $t = 0$, the initial conditions are :

$$\mathbf{u}(0) = \mathbf{u}_0 \,; \quad \tau(0) = \tau_0 \quad \text{in} \quad \Omega \,. \tag{2.9}$$

The incompressibility equation (2.4) requires the following compatibility condition :

$$\int_\Gamma \mathbf{u}_\Gamma \,.\, \mathbf{n}\, ds = 0 \tag{2.10}$$

where $ds$ is the measure on the boundary $\Gamma$.

### 2.4. Non-dimensional numbers and governing equations

In the present paper, added to the non-dimensional $Re$ and $We$ numbers, we use a retardation parameter $\alpha$ [14] given by :

$$\alpha = \frac{\eta_\nu}{\eta_n + \eta_\nu}. \tag{2.11}$$

Referring to the basic equations (2.2), (2.4)-(2.5) of the problem, we may now consider the following problem :

(P) : Find the non-dimensional quantities, still noted $\tau$, $\mathbf{u}$ and $p$, defined in $\Omega$, which verify the following equations :

$$We\left(\frac{\partial \tau}{\partial t} + (\mathbf{u}\,.\,\nabla)\,\tau + \beta_a(\tau, \nabla\mathbf{u})\right) + \tau - 2\,\alpha D(\mathbf{u}) = 0 \quad \text{in} \quad \Omega \tag{2.12}$$

$$Re\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\,.\,\nabla)\,\mathbf{u}\right) - \operatorname{div}\tau - (1-\alpha)\,\Delta\mathbf{u} + \nabla p = 0 \quad \text{in} \quad \Omega \tag{2.13}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{2.14}$$

subjected to boundary and initial conditions (2.7)-(2.9). The function $\beta_a$ involved in equation (2.12) is given by :

$$\beta_a(\tau, \nabla\mathbf{u}) = \tau\,.\,W(\mathbf{u}) - W(\mathbf{u})\,.\,\tau - a(D(\mathbf{u})\,.\,\tau + \tau\,.\,D(\mathbf{u}))\,. \tag{2.15}$$

### 2.5. Existence results

Some results concerning the existence of solutions of problem (P) (equations (2.12)-(2.14) and boundary condition (2.7)-(2.9)) are known. There results are obtained upon the assumptions made for the boundary $\Gamma$, the values $\tau_\Gamma$, $\mathbf{u}_\Gamma$, $\tau_0$ and $\mathbf{u}_0$ and data of the parameters $Re$, $We$ and $\alpha$ :

i) Renardy [21, 22] has obtained, using a fixed-point method, existence of stationary solutions for any value of $\alpha$, the other parameters being small.

ii) Guillopé and Saut [13, 14] have proved a global existence result of the solution of problem (P), in the unsteady case, for small values of $\alpha$.

## 3. SEMI-APPROXIMATION OF THE EVOLUTION USING A $\theta$-SCHEME

We now consider the problem (P) where the inertia term of equation (2.13) is ignored, since only slow flows are investigated. We present in this section a new method for solving problem (P) by the steps related to the alternating

direction implicit technique [20] for decoupling the two main difficulties, e.g., the non-linearity of equation (2.12) and the incompressibility equation (2.14). It will be shown that the corresponding algorithm enables to solve the problem under consideration in a reliable and efficient way.

### 3.1. The $\theta$-scheme

Let $H$ be a Hilbert space on $\mathbb{R}$. Consider a continuous operator $A$ on $H$ and the following problem :

Find $U \in L^{\infty}(H, \mathbb{R})$ such that :

$$m \frac{dU}{dt} + A(U) = 0 \tag{3.1}$$

$$U(0) = U_0 \tag{3.2}$$

given $U_0 \in H$ and $m \in \mathbb{R}$.

Using a decomposition of $A$ on the form :

$$A = A_1 + A_2 \tag{3.3}$$

we may associate the sequence $(U^n)_{n \geq 0}$, $U^n \in H$, defined by the following relations :

$$U(0) = U_0 \tag{3.4}$$

and, for $n \geq 0$, an implicit definition of $U^{n+1}$ according to the three-step following procedure :

$$m \frac{U^{n+\theta} - U^n}{\theta \, \Delta t} + A_1 U^{n+\theta} = - A_2 U^n \tag{3.5}$$

$$m \frac{U^{n+1-\theta} - U^{n+\theta}}{(1 - 2\theta) \Delta t} + A_2 U^{n+1-\theta} = - A_1 U^{n+\theta} \tag{3.6}$$

$$m \frac{U^{n+1} - U^{n+1-\theta}}{\theta \, \Delta t} + A_1 U^{n+1} = - A_2 U^{n+1-\theta} \tag{3.7}$$

where $\theta \in ]0, 1/2[$ and $\Delta t > 0$.

It should be pointed out that equation (3.7) corresponds to a symmetrization step, which allows us to ensure the stability of the $\theta$-scheme [11].

When $A_1$ and $A_2$ are linear operators, the scheme defined by equations (3.5)-(3.7) is unconditionally stable and has a error of $O(\Delta t^2)$.

### 3.2. Application to the solution of problem (P)

The relevant equations (2.12)-(2.14) of the slow flow of an Oldroyd-B fluid enable us to introduce two operators $A_1$ and $A_2$ related to the original

operator $A$ by (3.3) which may be defined as follows :

$$A_1(\tau, \mathbf{u}, p) = \begin{pmatrix} \dfrac{\omega}{2\alpha}\tau - D(\mathbf{u}) \\ \mathbf{div}\ \tau + (1-\alpha)\Delta\mathbf{u} - \nabla p \\ \mathrm{div}\ \mathbf{u} \end{pmatrix} \tag{3.8}$$

where $\Delta$ denotes the Laplace operator corresponding to a boundary condition of Dirichlet type (2.7), $\omega \in ]0, 1[$, and :

$$A_2(\tau, \mathbf{u}, p) = \begin{pmatrix} \dfrac{We}{2\alpha}((\mathbf{u}\,.\,\nabla)\tau + \beta_a(\tau, \nabla\mathbf{u})) + \dfrac{1-\omega}{2\alpha}\tau \\ 0 \\ 0 \end{pmatrix}. \tag{3.9}$$

The diagonal matrix involving real elements is :

$$m = \mathrm{diag}\left(\frac{We}{2\alpha}, -Re, 0\right). \tag{3.10}$$

Application of the procedure defined by equations (3.5)-(3.7) to operator (3.8) and (3.9) gives a new algorithm [24] which allows us to decouple of the computation of stresses and pressure-velocity (we have used the relations $\mathrm{div}\ \mathbf{u}^{n+\theta} = 0$ and $2\ \mathrm{div}\ D(\mathbf{u}^{n+\theta}) = \Delta\mathbf{u}^{n+\theta}$ in step 1). This algorithm is described below.

**Algorithm 3.1** alternating direction method $(We > 0)$

**step 1 :** $\tau^n$ and $\mathbf{u}^n$ being known, determine successively in an explicit way :

$$\gamma^n = (\mathbf{u}^n\,.\,\nabla)\tau^n + \beta_a(\tau^n, \nabla\mathbf{u}^n) \tag{3.11}$$

$$f_1 = \lambda\mathbf{u}^n + c_1\,\mathbf{div}\ \tau^n + c_2\,\mathbf{div}\ \gamma^n \tag{3.12}$$

then determine $(\mathbf{u}^{n+\theta}, p^{n+\theta})$ solution of $(S)$ :

$$\lambda\mathbf{u}^{n+\theta} - \eta\ \Delta\mathbf{u}^{n+\theta} + \nabla p^{n+\theta} = f_1 \quad \text{in} \quad \Omega \tag{3.13}$$

$$\mathrm{div}\ \mathbf{u}^{n+\theta} = 0 \quad \text{in} \quad \Omega \tag{3.14}$$

$$\mathbf{u}^{n+\theta} = u_\Gamma((n+\theta)\Delta t) \quad \text{on} \quad \Gamma \tag{3.15}$$

and compute :

$$\tau^{n+\theta} = c_1\,\tau^n + c_2\,\gamma^n + c_3\,D(\mathbf{u}^{n+\theta}) \tag{3.16}$$

**step 2 :** $\tau^{n+\theta}$, $\mathbf{u}^{n+\theta}$ and $\mathbf{u}^n$ being known, compute explicitly :

$$g^{n+1-\theta} = c_4\,\tau^{n+\theta} + c_5\,D(\mathbf{u}^{n+\theta}) \tag{3.17}$$

$$\mathbf{u}^{n+1-\theta} = \frac{1-\theta}{\theta}\,\mathbf{u}^{n+\theta} - \frac{1-2\theta}{\theta}\,\mathbf{u}^n \tag{3.18}$$

then find $\tau^{n+1-\theta}$ solution of $(T)$:

$$(\mathbf{u}^{n+1-\theta} \cdot \nabla)\, \tau^{n+1-\theta} + \beta_a(\tau^{n+1-\theta}, \nabla \mathbf{u}^{n+1-\theta}) + \nu \tau^{n+1-\theta} = g^{n+1-\theta} \tag{3.19}$$

$$\tau^{n+1-\theta} = \tau_\Gamma((n+1-\theta)\,\Delta t) \quad \text{on} \quad \Gamma_- \tag{3.20}$$

**step 3** is obtained by replacing $(n+\theta)$ by $n+1$, and $n$ by $(n+1-\theta)$ in step 1.

The different coefficients $\lambda$, $\eta$, $\nu$, $c_1$, $c_2$, $c_3$, $c_4$ and $c_5$ are obtained from $We$, $Re$, $\alpha$, $\theta$, $\Delta t$ and $\omega$ according to the following equations :

$$\lambda = \frac{Re}{\theta\, \Delta t} \tag{3.21}$$

$$\eta = 1 - \alpha\, \frac{We - (1-\omega)\,\theta\,\Delta t}{We + \omega\,\theta\,\Delta t} \tag{3.22}$$

$$\nu = \frac{1}{(1-2\,\theta)\,\Delta t} + \frac{1-\omega}{We} \tag{3.23}$$

$$c_1 = \frac{We - (1-\omega)\,\theta\,\Delta t}{We + \omega\,\theta\,\Delta t} \tag{3.24}$$

$$c_2 = -\frac{We\,\theta\,\Delta t}{We + \omega\,\theta\,\Delta t} \tag{3.25}$$

$$c_3 = \frac{2\,\alpha\,\theta\,\Delta t}{We + \omega\,\theta\,\Delta t} \tag{3.26}$$

$$c_4 = \frac{1}{(1-2\,\theta)\,\Delta t} - \frac{\omega}{We} \tag{3.27}$$

$$c_5 = \frac{2\,\alpha}{We}\,. \tag{3.28}$$

The algorithm 3.1 involves sub-problems $(S)$ of Stokes type and sub-problems $(T)$ of transport type. The steps 1 and 3 require the solving of sub-problems $(S)$, and remain well-posed [10] because $\eta > 0$ for all $\alpha \in [0, 1]$, $We > 0$ and $\Delta t > 0$. Especially for high Weissenberg numbers :

$$\lim_{We \to +\infty} \eta = 1 - \alpha > 0 \tag{3.29}$$

for all $\alpha \in [0, 1[$.

A sufficient condition of well-posedness of the sub-problem $(T)$ of transport type will be presented in section 5. Considering the choice of $\theta$, we have followed Glowinski [11] using $\theta = 1 - \frac{1}{\sqrt{2}}$.

## 4. APPROXIMATION WITH RESPECT TO SPACE

We now consider the steady flow problem $\left( \frac{\partial}{\partial t} = 0 \right)$ of the Oldroyd-B fluid. This problem can be regarded as a singular perturbation of the Stokes problem $(P_\alpha)$, obtained with $We = 0$ [22] :

$(P_\alpha)$ : find $\tau$, $\mathbf{u}$ and $p$, defined in $\Omega$, and such that :

$$\tau - 2\,\alpha D(\mathbf{u}) = 0 \quad \text{in} \quad \Omega \tag{4.1}$$

$$\mathbf{div}\,\tau + (1-\alpha)\,\Delta\mathbf{u} - \nabla p = -\mathbf{f} \quad \text{in} \quad H^{-1}(\Omega)^2 \tag{4.2}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{4.3}$$

$$\mathbf{u} = \mathbf{u}_\Gamma \quad \text{on} \quad \Gamma \tag{4.4}$$

where $\Omega$ is convex, $\alpha \in [0, 1]$, $\mathbf{f} \in H^{-1}(\Omega)^2$, $\mathbf{u}_\Gamma \in H^{1/2}(\Gamma)$ are given, and $\mathbf{u}_\Gamma$ verify (2.10).

The problem $(P_\alpha)$ is effectively equivalent to the Stokes problem $(P_0)$, of solution $(\mathbf{u}, p)$ independent of $\alpha$ :

$(P_0)$ : Find $\mathbf{u}$ and $p$, defined in $\Omega$, such that :

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad H^{-1}(\Omega)^2 \tag{4.5}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{4.6}$$

$$\mathbf{u} = \mathbf{u}_\Gamma \quad \text{on} \quad \Gamma\,. \tag{4.7}$$

Although equations (4.1)-(4.4) are still linear, it should be pointed out that great care is to be taken with the incompressibility condition (4.3). In the finite-element context, the use of classical approximation methods for the velocity field requires sophisticated elements, leading to more or less good approximation of equation (4.3). This is all the more troublesome as the velocity fiel $\mathbf{u}$ is likely to transport the stress tensor, for example in sub-problem $(T)$.

As the starting point of our analysis, we shall consider here the incompressible finite element of Thomas-Raviart [10, 23] for the velocities. Then, we shall list here the basic properties of this element. Finally, two possible approximations for the tensors which will assure the well-posedness of the related approximate problems are considered.

### 4.1. Approximation of vectors

We consider now a two-dimensional flow situation in a convex polygonal domain $\Omega$. In order to approximate the Stokes problem $(P_0)$, we introduce the (scalar) vorticity field $\omega$ as :

$$\omega = \operatorname{curl}\mathbf{u} = \partial_1 u_2 - \partial_2 u_1 \tag{4.8}$$

related to a vector field $\mathbf{u} = (u_1, u_2)$.

Using the notation :

$$\mathbf{curl}\ \omega = (\partial_2\omega\ ;\ -\partial_1\omega) \tag{4.9}$$

we may write the following equation :

$$\mathbf{curl}\ \mathrm{curl}\ \mathbf{u} = -\Delta\mathbf{u} + \nabla\ \mathrm{div}\ \mathbf{u} \tag{4.10}$$

Let $\Theta_h$, $V_h$ and $P_h$ be finite dimensional vector spaces such that :

$$\begin{aligned} &\Theta_h \subset H^1(\Omega) \\ &V_h \subset H(\mathrm{div}\ ;\ \Omega) \\ &P_h \subset L^2(\Omega)\,. \end{aligned} \tag{4.11}$$

We also consider the spaces :

$$V_{0h} = \{\mathbf{v} \in V_h\ ;\ \mathbf{v}\,.\,\mathbf{n} = 0 \quad \text{on} \quad \Gamma\} \tag{4.12}$$

and

$$P_{0h} = \left\{q \in L^2(\Omega)\ ;\ \int_\Omega q\ dx = 0\right\}. \tag{4.13}$$

An approximating variational formulation of problem $(P_0)$ involved by equations (4.5)-(4.7) is :

$(P_0)_h$ : Find $(\omega_h, \mathbf{u}_h, p_h) \in \Theta_h \times V_h \times P_{0h}$ such that $\mathbf{u}\,.\,\mathbf{n} = \mathbf{u}_\Gamma\,.\,\mathbf{n}$ on $\Gamma$ and :

$$(\omega_h, \theta) - (\mathbf{curl}\ \theta, \mathbf{u}_h) = \langle \mathbf{u}_\Gamma\,.\,\mathbf{t}, \theta\rangle \tag{4.14}$$

$$-(\mathbf{curl}\ \omega_h, \mathbf{v}) + (\mathrm{div}\ \mathbf{v}, p_h) = -(\mathbf{f}, \mathbf{v}) \tag{4.15}$$

$$(\mathrm{div}\ \mathbf{u}_h, q) = 0 \tag{4.16}$$

for any $(\theta, \mathbf{v}, q) \in \Theta_h \times V_{0h} \times P_h$. We suppose here $\mathbf{f} \in L^2(\Omega)^2$. The symbols $(\,.\,,\,.\,)$ and $\langle\,.\,,\,.\,\rangle$ denote the scalar product of spaces $L^2(\Omega)$ and $L^2(\Gamma)$ respectively. The vector $\mathbf{t}$ is written as : $\mathbf{t} = (-n_2, n_1)$ if $\mathbf{n} = (n_1, n_2)$.

Lastly, we define the spaces $X_h$ and $X_{0h}$ by the following expressions :

$$X_h = \{\mathbf{v} \in V_h\ ;\ (\mathrm{div}\ \mathbf{v}, q) = 0,\ \text{for all}\ q \in P_h\} \tag{4.17}$$

$$X_{0h} = \{\mathbf{v} \in X_h\ ;\ \mathbf{v}\,.\,\mathbf{n} = 0 \quad \mathbf{on} \quad \Gamma\}\,. \tag{4.18}$$

On the basis on [10], if the spaces $\Theta_h$, $V_h$ and $P_h$ satisfy the following compatibility conditions :

$$X_h \subset \mathbf{curl}\ \Theta_h \tag{4.19}$$

$$(q_h \in P_h \text{ and } (\mathrm{div}\ \mathbf{v}_h, q_h) = 0,\ \text{for all}\ \mathbf{v}_h \in X_{0h}) \Rightarrow q_h = 0 \tag{4.20}$$

then the problem $(P_0)_h$ (equations (4.14)-(4.16)) has a unique solution.

Practically, following [10, chap. 3] and [23], we adopt the following spaces :

$$\Theta_h = \left\{\theta \in H^1(\Omega) \cap C^0(\Omega)\,;\, \theta_{|K} \in P_k(K), \text{ for all } K \in T_h\right\} \quad (4.21)$$

$$V_h = \left\{\mathbf{v} \in H\ (\text{div}\,;\,\Omega)\,;\, \mathbf{v}_{|K} \in RT_k(K), \text{ for all } K \in T_h\right\} \quad (4.22)$$

$$P_h = \left\{q \in L^2(\Omega)\,;\, q_{|K} \in P_{k-1}(K), \text{ for all } K \in T_h\right\} \quad (4.23)$$

which satisfy the conditions (4.19)-(4.20). The subspace $RT_k(K)$ is defined hereafter. $(T_h)_{h>0}$ denotes a family of regular triangulations of $\bar{\Omega}$ indexed by a parameter $h > 0$. $T_h$ is a triangulation composed of triangles or convex quadrilaterals of diameter majored by $h$. $k \geqslant 1$ denotes an integer which is used to define the level of the finite element method. $P_l(K)$, $l \geqslant 0$, denotes the subspace of dimension $\frac{1}{2}(l+1)(l+2)$ related to polynomials defined on a triangle $K$ (respectively $(l+1)^2$ for a convex quadrilateral) of degree lower or equal than $l$ with respect to the two variables (respectively, to each variable).

These definitions imply that elements of $\Theta_h$ are continuous at the interfaces of the elements, which is not verified, in general, for elements of $P_h$. According to the definition (4.22) of $V_h$, the assumption $\mathbf{v} \in H\ (\text{div}\,;\,\Omega)$ is equivalent to verify the continuity of normal components of $\mathbf{v} \in V_h$ through the interfaces of the elements.

We define the subspace $RT_k(K)$ by :

$$RT_k(K) = \left\{\mathbf{v} \in H\ (\text{div}\,;\,K)\,;\, J_K^{-1} \,.\, DF_K(\mathbf{v}) \in RT_k(\hat{K})\right\} \quad (4.24)$$

where $F_K$ is the invertible application which maps the triangle (respectively square) $\hat{K}$ of reference into a triangle (respectively convex quadrilateral) $K$. $DF_K$ denotes the Jacobian matrix of $K$ and $J_K$ its determinant.

When $\hat{K}$ is the reference triangle, $RT_k(\hat{K})$ is found to be the space of dimension $k(k+2)$ of vectors $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2)$ of the form :

$$\hat{v}_1(\xi_1, \xi_2) = \hat{p}_1(\xi_1, \xi_2) + \left(\sum_{i=0}^{k-1} \alpha_i\, \xi_1^{k-i-1}\, \xi_2^i\right) \cdot \xi_1 \quad (4.25)$$

$$\hat{v}_2(\xi_1, \xi_2) = \hat{p}_2(\xi_1, \xi_2) + \left(\sum_{i=0}^{k-1} \alpha_i\, \xi_1^{k-i-1}\, \xi_2^i\right) \cdot \xi_2 \quad (4.26)$$

where $\hat{p}_1,\ \hat{p}_2 \in P_{k-1}(\hat{K})$ and $\alpha_i \in \mathbb{R}$, $0 \leqslant i \leqslant k-1$. We also get :

$$\text{div}\ \hat{\mathbf{v}} = 0 \quad \text{if and only if} \quad \alpha_i = 0, \text{ for all } i,\ 0 \leqslant i \leqslant k-1\,. \quad (4.27)$$

In the case where $\hat{K}$ is the reference square $[-1, 1]^2$, it can be shown that $RT_k(\hat{K})$ is the space of dimension $2k(k+1)$ of vectors $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2)$ of the form :

$$\hat{v}_1 = \hat{p}_1(\xi_2) + \hat{q}_1(\xi_1, \xi_2) \cdot \xi_1 \tag{4.28}$$

$$\hat{v}_2 = \hat{p}_2(\xi_1) + \hat{q}_2(\xi_1, \xi_2) \cdot \xi_2 \tag{4.29}$$

where $\hat{p}_1, \hat{p}_2 \in P_{k-1}([0, 1])$ and $\hat{q}_1, \hat{q}_2 \in P_{k-1}(\hat{K})$. It is immediately seen that :

$$\text{div } \hat{\mathbf{v}} = 0 \text{ if and only if } \partial_1(\xi_1 \cdot \hat{q}_1) + \partial_2(\xi_2 \cdot \hat{q}_2) = 0 . \tag{4.30}$$

*Remark 4.1 :* If $T_h$ involves only triangles, the property (4.27) may be concisely written as :

$$X_h = P_h^2 . \tag{4.31}$$

*Remark 4.2 :* If $T_h$ involves only rectangles, the sides of which are parallel to the reference axes, we obtain the interesting following property : for each $\mathbf{v} = (v_1, v_2) \in X_{0h}$, there exists $\psi_1, \psi_2 \in P_h$ such that :

$$v_1(x_1, x_2) = x_1 \cdot \psi_1(x_1, x_2) \quad \text{and} \quad v_2(x_1, x_2) = x_2 \cdot \psi_2(x_1, x_2) \tag{4.32}$$

and

$$\partial_1(x_1 \cdot \psi_{1|K}) + \partial_2(x_2 \cdot \psi_{2|K}) = 0 \text{ in } K , \quad \text{for all} \quad K \in T_h . \tag{4.33}$$

### 4.2. Approximation of tensors

We now consider the problem $(P_1)$ resulting from equations (4.1)-(4.4) for $\alpha = 1$. By relaxing the symmetry constraint $\tau = \tau^t$ of the stress tensor, we define a family of finite elements which leads to a well-posed problem. Then, introducing a family of triangulations constituted by rectangles the sides of which are parallel to the axes, we propose a symmetric approximation of the tensors that generalizes the « Marker-and-Cell » [25] technique.

#### 4.2.1. *Relaxation of the symmetry constraint*

Let $T_h$ be a finite-dimensional subspace which verifies :

$$T_h \subset \left\{ \gamma = (\gamma_{ij}) ; (\gamma_{i1}, \gamma_{i2}) \in H(\text{div} ; \Omega), i = 1, 2 \right\} . \tag{4.34}$$

We now introduce a Lagrange multiplier $\lambda$, which belongs to a finite-dimensional space $M_h \subset L^2(\Omega)$ and define the space $S_h$ by :

$$S_h = \left\{ \gamma \in T_h ; (\gamma_{12} - \gamma_{21}, \mu) = 0 , \quad \text{for all} \quad \mu \in M_h \right\} . \tag{4.35}$$

An approximate variational formulation of the problem $(P_1)$ may be expressed as follows :

$(P_1)_h$ : find $(\tau_h, \mathbf{u}_h, \lambda_h, p_h) \in T_h \times V_h \times M_h \times P_{0h}$, $\mathbf{u} \cdot \mathbf{n} = \mathbf{u}_\Gamma \cdot \mathbf{n}$ on $\Gamma$, such that :

$$\frac{1}{2}(\tau_h, \gamma) + (\mathbf{div}\ \gamma, \mathbf{u}_h) + (\gamma_{12} - \gamma_{21}, \lambda_h) = \langle \mathbf{u}_\Gamma, \gamma \cdot \mathbf{n} \rangle \quad (4.36)$$

$$(\mathbf{div}\ \tau_h, \mathbf{v}) + (\operatorname{div} \mathbf{v}, p_h) = -(\mathbf{f}, \mathbf{v}) \quad (4.37)$$

$$(\tau_{h;12} - \tau_{h;21}, \mu) = 0 \quad (4.38)$$

$$(\operatorname{div} \mathbf{u}_h, q) = 0 \quad (4.39)$$

for any $(\gamma, \mathbf{v}, \mu, q)$ in $T_h \times V_{0h} \times M_h \times P_h$.

PROPOSITION 4.1 : *If the spaces $T_h$, $V_h$, $M_h$ and $P_h$ verify the compatibility conditions* (4.20) *and :*

if $(\mathbf{v}_h, \mu_h) \in X_{0h} \times M_h$,

$$(\mathbf{div}\ \gamma, \mathbf{v}_h) + (\gamma_{12} - \gamma_{21}, \mu_h) = 0\,, \quad \text{for all} \quad \gamma \in T_h \quad (4.40)$$

*then* $\mathbf{v}_h = 0$ *and* $\mu_h = 0$.

Then, the problem $(P_1)_h$ admits a unique solution.

*Proof :* Equations (4.36)-(4.39) of $(P_1)_h$ define a closed linear system, it is sufficient to prove the uniqueness property.

Suppose $\mathbf{f} = 0$ and $\mathbf{u}_\Gamma = 0$, and let $(\tau_h, \mathbf{u}_h, \lambda_h, p_h)$ a solution of the problem under consideration. Putting (4.39) with $q = p_h$ in equation (4.37) with $\mathbf{v} = \mathbf{u}_h$, we get $(\mathbf{u}_h, \mathbf{div}\ \tau_h) = 0$. Similarly, using equation (4.38) with $\mu = \lambda_h$ in equation (4.36) with $\gamma = \tau_h$, leads to $\tau_h = 0$. Thus, equation (4.36) may be written as :

$$(\mathbf{div}\ \gamma, \mathbf{u}_h) + (\gamma_{12} - \gamma_{21}, \lambda_h) = 0$$

for all $\gamma \in T_h$, that means, according to (4.40) : $\mathbf{u}_k = 0$ and $\lambda_h = 0$. In that case, with the help of (4.20), equation (4.37) implies that $p_h = 0$. □

We now select the following spaces :

$$T_h^{(1)} = \left\{ \gamma = (\gamma_{ij}) \in H(\operatorname{div} ; \Omega) ; (\gamma_{i1}, \gamma_{i2}) \in V_h,\ i = 1, 2 \right\} \quad (4.41)$$

$$M_h^{(1)} = P_h \quad (4.42)$$

and $S_h^{(1)}$, related to definition (4.35).

THEOREM 4.1 : *When the subspaces $T_h^{(1)}$ (equation* (4.41)*), $V_h$* (4.22)*, $M_h^{(1)}$* (4.42) *and $P_h$* (4.23) *are selected as spaces $T_h$, $V_h$, $M_h$, $P_h$, respectively, the problem $(P_1)_h$ admits a unique solution.*

*Proof :* It is sufficient to verify condition (4.40). For purpose of simplification, let us suppose that the triangulation $T_h$ involves only triangles.

First, we consider $\gamma \in T_h^{(1)}$ such that div $\gamma = 0$. According to (4.31), we have $\gamma \in P_h^4$. With $\gamma_{11} = \gamma_{22} = 0$ and $\gamma_{12} = -\gamma_{21} = \frac{1}{2}\mu_h \in P_h$, we get $\mu_h = 0$.

Secondly, using (4.40) for every component of $\mathbf{v}_h$, we obtain, from (4.20) : $\mathbf{v}_h = 0$. □

*Remark 4.3 :* The element of lowest order $k = 1$ and the nodal localization of degrees of freedom are shown on figure 1.

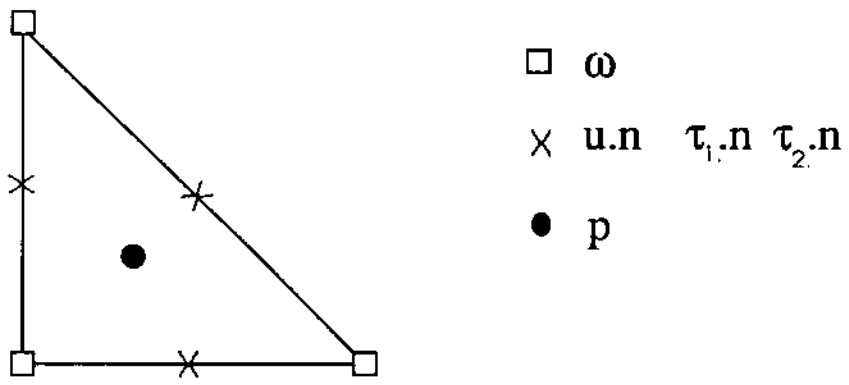


**Figure 1. — Triangular element ($k = 1, 2$).**

4.2.2. *Symmetric approximation of the tensors*

When the triangulation $T_h$ only involves rectangles the sides of which are parallel to the reference axes, it is possible to define the derivatives $\partial_i \nu_i$, $i = 1, 2$ as elements of $P_h$, for a given element $\mathbf{v} = (\nu_1, \nu_2) \in V_h$. The main idea consists in considering separately the normal components $\tau_{ii}$, $i = 1, 2$ and the shearing components $\tau_{ij} = \tau_{ji}$, $i \neq j$ of the stress tensor $\tau = (\tau_{ij})$.

Let $S_h$ be a finite-dimensional subspace defined as :

$$S_h \subset \left\{ \gamma = (\gamma_{ij}) ; \gamma_{ii} \in L^2(\Omega) ; \gamma_{ij} = \gamma_{ji} \in H^1(\Omega) \cap C^0(\Omega),\ i \neq j \right\} . \tag{4.43}$$

An approximate variational formulation of problem $(P_1)$ may be expressed as :

$(\tilde{P}_1)_h$ : find $(\tau_h, \mathbf{u}_h, p_h) \in S_h \times V_h \times P_{0h}$, $\mathbf{u} = \mathbf{u}_\Gamma$ on $\Gamma$, such that

$$\frac{1}{2}(\tau_h, \gamma) - \sum_{i=1}^{2} (\gamma_{ii}, \partial_i \nu_i) + \sum_{i \neq j} (\partial_j \gamma_{ij}, u_{h;i}) = \sum_{i \neq j} \langle \gamma_{ij} \cdot n_j, u_{\Gamma;i} \rangle \tag{4.44}$$

$$-\sum_{i=1}^{2}(\tau_{h;ii},\partial_i\nu_i)+\sum_{i\neq j}(\partial_j\tau_{h;ij},\nu_i)+$$

$$+(\operatorname{div}\mathbf{v},p_h)=-(\mathbf{f},\mathbf{v})-\sum_{i=1}^{2}\langle\tau_{\Gamma;ii}\cdot n_i,\nu_i\rangle \quad (4.45)$$

$$(\operatorname{div}\mathbf{u}_h,q)=0 \quad (4.46)$$

for any $(\gamma,\mathbf{v},q)\in S_h\times V_{0h}\times P_h$, where $\tau_{\Gamma;ii}=2\,\partial_i\mathbf{u}_{\Gamma;i}$ on $\Gamma$.

PROPOSITION 4.2 : *If the spaces $S_h$, $V_h$ and $P_h$ verify the compatibility condition* (4.20) *and :*

if $\mathbf{v}_h\in X_{0h}$,

$$-\sum_{i=1}^{2}(\gamma_{ii},\partial_i\nu_{i;h})+\sum_{i\neq j}(\partial_j\gamma_{ij},\nu_{i;h})=0\,,\quad\text{for all}\quad\gamma\in S_h \quad (4.47)$$

*then* $\mathbf{v}_h=0$

*then a unique solution for problem* $(\tilde{P}_1)_h$ *exists.*

*Proof :* The proof is similar to that given for proposition 4.1. □

We now consider the space :

$$S_h^{(2)}=\{\gamma=(\gamma_{ij})\,;\,\gamma_{ii}\in P_h\,;\,\gamma_{ij}=\gamma_{ji}\in\Theta_h\}\,. \quad (4.48)$$

THEOREM 4.2 : *If $S_h^{(2)}$ is considered as space $S_h$, and $V_h$ and $P_h$ are given by* (4.22) *and* (4.23), *then the problem $(P_1)_h$ admits a unique solution.*

*Proof :* We only have to verify the compatibility condition (4.47). Let $\mathbf{v}_h\in X_{0h}$. From remark 4.2, there exist $\psi_1$, $\psi_2\in P_h$ such that $\mathbf{v}_h=(x_1\,\psi_1,x_2\,\psi_1)$. A choice of $\gamma\in S_h$ which satisfy $\gamma_{11|K}=\partial_1(x_1\cdot\psi_{1|K})$ and $\gamma_{22|K}=\partial_2(x_2\cdot\psi_{2|K})$ for all $K\in T_h$, $\gamma_{12}=\gamma_{21}=0$ leads to $\partial_1(x_1\cdot\psi_{1|K})=\partial_2(x_2\cdot\psi_{2|K})=0$ in $K$, for all $K\in T_h$. Hence $\psi_1=\psi_2=0$ : $\mathbf{v}_h=0$. □

*Remark 4.4 :* In figure 2, we present a rectangular element the sides of which are parallel to the axes. The element of lowest order $k=1$ may be considered as an extension of the « Marker and Cell » scheme [25].

### 4.3. Approximation of the Stokes problem

From the elements presented in previous sections we propose a variational approximation of problem (4.1)-(4.4) :

$(P_\alpha)_h$ : Find $(\tau_h,\omega_h,\mathbf{u}_h,p_h)\in S_h\times\Theta_h\times V_h\times P_{0h}$, $\mathbf{u}_h\cdot\mathbf{n}=\mathbf{u}_\Gamma\cdot\mathbf{n}$ on $\Gamma$, such that :

$$(\tau_h,\gamma)+2\,\alpha\,(\operatorname{div}\gamma,\mathbf{u}_h)=2\,\alpha\,\langle\mathbf{u}_\Gamma,\gamma\cdot\mathbf{n}\rangle \quad (4.49)$$

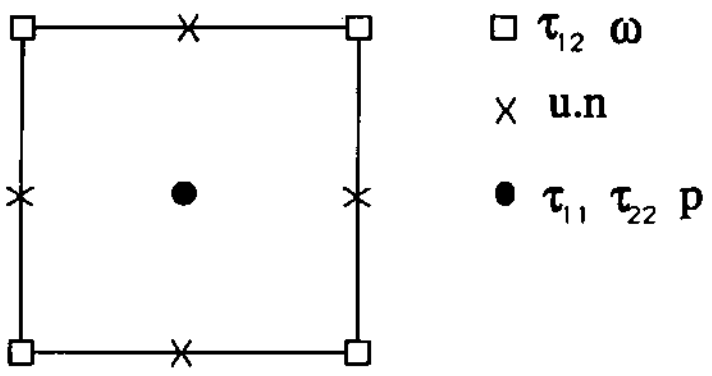


Figure 2. — Rectangular element ($k = 1, 2$).

$$(\omega, \theta) - (1 - \alpha)(\mathbf{curl}\ \theta, \mathbf{u}_h) = (1 - \alpha)\langle \mathbf{u}_\Gamma \cdot \mathbf{t}, \theta \rangle \tag{4.50}$$

$$(\mathbf{div}\ \tau_h, \mathbf{v}) - (\mathbf{curl}\ \omega_h, \mathbf{v}) + (p_h, \mathbf{div}\ \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) \tag{4.51}$$

$$(\mathrm{div}\ \mathbf{u}_h, q) = 0 \tag{4.52}$$

for each $(\gamma, \theta, \mathbf{v}, q) \in T_h \times \Theta_h \times V_{0h} \times P_h$.

Moreover, when the finite spaces are expressed by (4.21)-(4.23) and (4.41) or (4.48), it can be readily proved from the previous results that the problem $(P_\alpha)_h$ admits a unique solution, for all $\alpha \in [0, 1]$. When $S_h$ is given by (4.48), $(\mathbf{u}_h, p_h)$ is independent of $\alpha$.

The algebraic equations involved by equations (4.49)-(4.52) may be written as :

$$\begin{pmatrix} M_\tau & 0 & 2\alpha B' & 0 \\ 0 & M_\omega & (1-\alpha)R' & 0 \\ B & R & 0 & D' \\ 0 & 0 & D & 0 \end{pmatrix} \begin{pmatrix} T \\ W \\ U \\ P \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix} \tag{4.53}$$

where $T$, $W$, $U$, $P$ denote the vectors of degrees of freedom related to $\tau_h$, $\omega_h$, $\mathbf{u}_h$ and $p_h$, respectively.

Using appropriate quadrature formulas for the evaluation of the coefficients of $M_\tau$ and $M_\omega$, the matrices may be diagonalized. Then, the vector $W$ can be easily eliminated from the system, by defining :

$$C = -(1 - \alpha) R \cdot M_\omega^{-1} \cdot R^t . \tag{4.54}$$

Hence, we obtain :

$$\begin{pmatrix} M_\tau & 2\alpha B' & 0 \\ B & C & D' \\ 0 & D & 0 \end{pmatrix} \begin{pmatrix} T \\ U \\ P \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} . \tag{4.55}$$

By introducing the transport terms for the stresses, in order to involve the viscoelastic effects, the relevant system is written as :

$$\begin{pmatrix} A(U) & 2\,\alpha B^t & 0 \\ B & C & D^t \\ 0 & D & 0 \end{pmatrix} \begin{pmatrix} T \\ U \\ P \end{pmatrix} = \begin{pmatrix} \tilde{F}_1 \\ F_2 \\ F_3 \end{pmatrix} \tag{4.56}$$

where $A(U) = M_\tau + We\,.\,T(U)$ and the « perturbation » matrix $T(U)$ is obtained from approximation of the transport term $(\mathbf{u}\,.\,\nabla)$. The approximating techniques are considered in the following section.

## 5. APPROXIMATION FOR THE TRANSPORT OF STRESSES

In this section, we present the analysis for approximation of sub-problems $(T)$ related to algorithm 3.1 :

$(T)$ : Given $\nu > 0$, $\mathbf{u} \in W^{1,\infty}(\Omega)^2$ and $g \in L^2(\Omega)^{4_s}$, find $\tau$, defined in $\Omega$, which verify :

$$(\mathbf{u}\,.\,\nabla)\,\tau + \beta_a(\tau, \nabla u) + \nu\tau = g \quad \text{in} \quad \Omega \tag{5.1}$$

$$\tau = \tau_\Gamma \quad \text{on} \quad \Gamma_- \,. \tag{5.2}$$

The bilinear form $\beta_a$ is given by equation (2.15).

In order to avoid numerical oscillating solutions which may appear despite the existence of a regular exact solution, we have selected methods which permit to approximate the transport operator $(\mathbf{u}\,.\,\nabla)$ by a monotonous (or total variation decreasing) operator.

A sufficient condition to existence of weak solutions to problem (5.1)-(5.2) is given in [24] :

$$\nu - 2\,|a|\,\|D(\mathbf{u})\|_\infty > 0 \tag{5.3}$$

where $a$ denotes the parameter involved in the objective derivation operator, and $\|\,.\,\|_\infty$ the norm of $L^\infty(\Omega)^4$.

For $a \neq 0$, and $\nu$ given by (3.23), a sufficient condition is :

$$\Delta t < \frac{1}{2\,|a|\,(1 - 2\,\theta)\,\|D(\mathbf{u})\|_\infty} \,. \tag{5.4}$$

In the following, upwind schemes for the approximating spaces will be examined.

### 5.1. Non-symmetric tensors

To simplify the notation, we will consider the approximation of the transport term for a general second-order tensor. The reader has to take care to introduce the symmetry constraint and the corresponding Lagrange multiplier for the problem under consideration.

Components of a tensor $\tau_h \in T_h^{(1)}$ are generally discontinuous at interfaces between elements (more precisely, the tangential components $\tau_i \cdot \mathbf{t}_K$, $i = 1, 2$ are discontinuous through the elements interfaces). Non-centered techniques are possible in schemes involving right-side or left-side values, according to the direction of the flow [17]. Then, the approximate problem may be written as follows :

$(T)_h$ : find $\tau_h \in T_h^{(1)}$, $\tau_h = \tau_\Gamma$ on $\Gamma_-$ such that

$$\sum_{K \in T_h} \left( - \int_{\partial K_-} (\mathbf{u} \cdot \mathbf{n}_K)[\tau_h] \cdot \gamma \, ds + \int_K (\mathbf{u} \cdot \nabla) \tau_h \cdot \gamma \, dx \right) + \int_\Omega (\beta_a(\tau_h, \nabla \mathbf{u}) + \nu \tau_h) \cdot \gamma \, dx = \int_\Omega g \cdot \gamma \, dx \quad (5.5)$$

for each $\gamma \in T_h^{(1)}$, $\gamma = 0$ on $\Gamma_-$.

We note $[\tau_h] = \tau_h^{\text{int}} - \tau_h^{\text{ext}}$ the step of $\tau_h$ at the separating line between one element to another in the direction of the outer unit normal vector $\mathbf{n}_K$ to the element $K$. We define :

$$\partial K_- = \{ x \in \partial K \, ; \, (\mathbf{u} \cdot \mathbf{n}_K)(x) < 0 \} \, . \quad (5.6)$$

The numerical analysis of this scheme has been extensively described by Girault and Raviart [10] for Navier-Stokes equations.

### 5.2. Symmetric tensors

In problem $(T)$, a scalar transport problem may be considered for each component of tensor $\tau$.

The normal component $\tau_{h;ii}$ are generally discontinuous and the method previously presented in section 5.1 can be applied in order to obtain the non-centered corresponding scheme.

The shear components $\tau_{h;ij}$, $i \neq j$ are continuous. In that case, the streamline upwind scheme as proposed by Hughes and Brooks [3] may be considered. The « non-consistent » formulation of the streamline upwind technique for the shear components is considered here. (See [18] for a comparison of efficiency with the Petrov-Galerkin formulation).

It should be pointed out that :

• The discontinuous method leads to monotonous schemes [12] only for element whose degree is equal or lower than 1 ;

• Although the streamline upwinding technique is not monotonous, the corresponding numerical results are generally found to be acceptable in many cases of practical interest.

## 6. NUMERICAL EXPERIMENTS

We now describe the application of the ADI method (already presented in 3.1) to the computation of the flow in a plane or axisymmetric contraction, subjected to specified boundary conditions. The results were obtained for different meshes, at various Weissenberg numbers.

### 6.1. General features of the flow in an abrupt contraction

#### 6.1.1. *Introduction*

Such a flow is of interest from both theoretical and practical points of view (e.g., in relation to polymer processing problems), and has been investigated in numerous experimental works (see for example [2] and [6]). In the entry flow region, before the contraction, the fluid particles are accelerated close to the central region of the duct, while vortices appear close to the edges (see *fig.* 3). This flow may be considered a complex one, because it appears to be practically a shear flow close to the solid wall, and becomes rather elongational at the vicinity of the center. Experiments reported in the literature have shown increasing recirculating zones when the value of the Weissenberg number $We$ increases.

#### 6.1.2. *Boundary conditions*

In our numerical experiments, the abrupt contraction was considered as plane (plane flow, with cartesian coordinates $x_1 = x$, $x_2 = y$), with a plane of symmetry for $x = 0$, or axisymmetrical (we use cylindrical coordinates $x_1 = z$, $x_2 = r$, $x_3 = \theta$), involving an axis of symmetry $r = 0$. In both situations, we may consider a half-domain $\Omega$ corresponding to $x_2 > 0$, as shown in figure 4.

The rheological model of equation (1.3) involves the upper convected derivative for $a = 1$. The computational domain is assumed to be long enough to verify Poiseuille velocity profiles at upstream and downstream sections $x_1 = s_1$ and $x_2 = s_2$ of respective widths $r_1$ and $r_2$. Then we have :

$$u_{\Gamma;1}(s_i, x_2) = \frac{f_i}{2}\left(1 - \left(\frac{x_2}{r_i}\right)^2\right), \quad 0 \leqslant x_2 \leqslant r_i, \quad i = 1, 2 \qquad (6.1)$$

Figure 3. — Flow description in an abrupt contraction.

Figure 4. — Domain $\Omega$ for an abrupt contraction $C = 4$.

$$u_{\Gamma;2}(s_i, x_2) = 0\,, \quad 0 \leqslant x_2 \leqslant r_i, \quad i = 1, 2 \tag{6.2}$$

where $f_i \in \mathbb{R}$, $i = 1, 2$.

For points in the plane or the axis of symmetry, we may write the following symmetry equation :

$$\frac{\partial u_{\Gamma;1}}{\partial x_2}(x_1, 0) = u_{\Gamma;2}(x_1, 0) = 0\,, \quad s_1 \leqslant x_1 \leqslant s_2 \tag{6.3}$$

and the boundary condition for the velocity :

$$\mathbf{u}_\Gamma = 0 \quad \text{at the wall}\,. \tag{6.4}$$

From the compatibility equation (2.10) related to the mass conservation, we get :

$$f_1 = \begin{cases} C^{-1} f_2 & \text{for a plane domain} \\ C^{-2} f_2 & \text{for an axisymmetric domain} \end{cases} \tag{6.5}$$

where $C = r_1/r_2$ denotes the contraction ratio.

The stresses at the upstream section are given by :

$$\tau_{11}(s_1, x_2) = 2\,\alpha We\,\frac{f_1^2 x_2^2}{r_1^4}\,, \quad 0 \leqslant x_2 \leqslant r_1 \tag{6.6}$$

$$\tau_{22}(s_1, x_2) = 0\,, \quad 0 \leqslant x_2 \leqslant r_1 \tag{6.7}$$

$$\tau_{12}(s_1, x_2) = -\,\alpha\,\frac{f_1 x_2}{r_1^2}\,, \quad 0 \leqslant x_2 \leqslant r_1\,. \tag{6.8}$$

In the axisymmetric case, the component $\tau_{33}$ (related to the coordinate $\theta$) may be computed, and verify at the upstream section the following equation :

$$\tau_{33}(s_1, x_2) = 0\,, \quad 0 \leqslant x_2 \leqslant r_1\,. \tag{6.9}$$

The choice :

$$f_2 = r_2 \tag{6.10}$$

led us to fix the shear component $\sigma_{12}$ of the Cauchy stress tensor $\sigma$ (equation (2.1)) at the wall, in the downstream fully developed flow : $|\sigma_{12}| \to 1$ when $s_2 \to +\infty$. This choice allows us to fix the average velocity at the downstream to 1/3 (resp. 1/8) for a plane (resp. axisymmetric) domain.

## 6.2. Triangulation and finite elements

### 6.2.1. *Choice of parameters*

In our numerical investigations, we have chosen $C = 4$ and $r_2 = 1$. This corresponds to the classical four-to-one abrupt contraction generally investigated in numerical works. The values $s_1 = -64$ and $s_2 = 200$, related to positioning the upstream and downstream sections have proved to be satisfactory for obtaining accurate Poiseuille velocity profiles. The parameter $\alpha$ of the Oldroyd-B model was taken to be $\alpha = 8/9$, as usually done in calculations. The range of the Weissenberg numbers was : $0 \leqslant We \leqslant 90$. The tests were run on a Appolo DN 4000 workstation.

### 6.2.2. *Mesh and elements*

The choice of rectangular finite elements has proved to be well-adapted to the shape of the computational flow domain. We have retained the elements of lower degree ($k = 1$) with a symmetric approximation for the stresses (see section 4.2.2).

Starting from a rough triangulation $T^{(0)}$, a family $(T^{(i)})_{i \geqslant 0}$ was constructed by using a refining procedure of $T^{(0)}$ near the re-entrant corner (*fig.* 5). On $T^{(i)}$, the length of the elements close to the re-entrant corner is given by :

$$h_i = \frac{1}{3 \times 2^i} . \tag{6.11}$$

The length of elements quite distant from the corner is given by a geometric progression of constant factor. $NE_i$ and $NS_i$ denote the number of elements and vertices, respectively, such that :

$$N_i = \dim\ (S_h \times V_h \times P_h) = 4\,NE_i + 2\,NS_i - 1 . \tag{6.12}$$

It can be noticed that $NE_i \backsim NS_i$ asymptotically, which leads to the approximation $N_i/NE_i \approx 6$ (see table 1).

**Table 1. — Number of elements of the triangulations.**

| $i$ | $h_i$ | $NF_i$ | $NS_i$ | $N_i$ | $N_i/NF_i$ |
|---|---|---|---|---|---|
| 0 | 0.3333 | 144 | 184 | 945 | 6.56 |
| 1 | 0.1667 | 299 | 356 | 1 909 | 6.38 |
| 2 | 0.0833 | 576 | 655 | 3 615 | 6.27 |
| 3 | 0.0416 | 1 196 | 1 309 | 7 403 | 6.18 |

**Figure 5. — Mesh family of the domain $\Omega$ (partial views).**

Consequently, the method uses asymptotically a degree of freedom by element and by (one of the 6) scalar fields $\tau_{11}$, $\tau_{12}$, $\tau_{22}$, $u_1$, $u_2$, and $p$, which appear to be an optimal number of unknowns.

### 6.3. Resolution of the sub-problems

The sub-problems $(S)$ of the Stokes-type are solved using a conjugate-gradient method. The preconditioning is obtained by means of augmented Lagrangian methods [8]. The efficiency of this method was particularly spectacular, as the initial problem is ill-conditioned, due to the domain length and the presence of the re-entrant corner.

The transport sub-problems $(T)$ are solved by using SSOR with block symmetrization [24]. All the elements are numbered in the direction of the main flow. Numerical tests of this procedure have shown fast convergence (in iteration number and time) of the residual terms.

### 6.4. Results for a four-to-one abrupt contraction

We now consider the numerical results obtained for the plane or axisymmetric contraction, in relation to the presence of the corner, which is expected to generate important vortices near the salient corner, and modifications on the evolution of stresses and velocities.

#### 6.4.1. *Normal stress component $\tau_{11}$ at the vicinity of the corner*

We first observe on figure 6 the influence of the Weissenberg number on the first normal stress $\tau_{11}$ along the straight line $x_2 = 1$ close to the re-entrant corner. We notice that :

(i) the peak (theoretically infinite at the corner singularity) becomes sharper and higher as the Weissenberg number increases. Similar examples could be also presented for the other stress components $\tau_{12}$ and $\tau_{22}$,

(ii) the accuracy of our computations can be asserted by considering the fully-developed Poiseuille profiles (for velocity and stress) at the downstream section. The shortest downstream section length such that the Poiseuille profiles are fully-developed increases as the Weissenberg number becomes greater (see also [4, 7]).

The stress at the downstream section tends to $2\,\alpha We\,\dfrac{f_2^2}{r_2^2}$.

#### 6.4.2. *Recirculating zones*

In order to comment the numerical results for the vortices, we have computed the stream function $\psi$, which is characterized as the unique solution of the following variational problem :

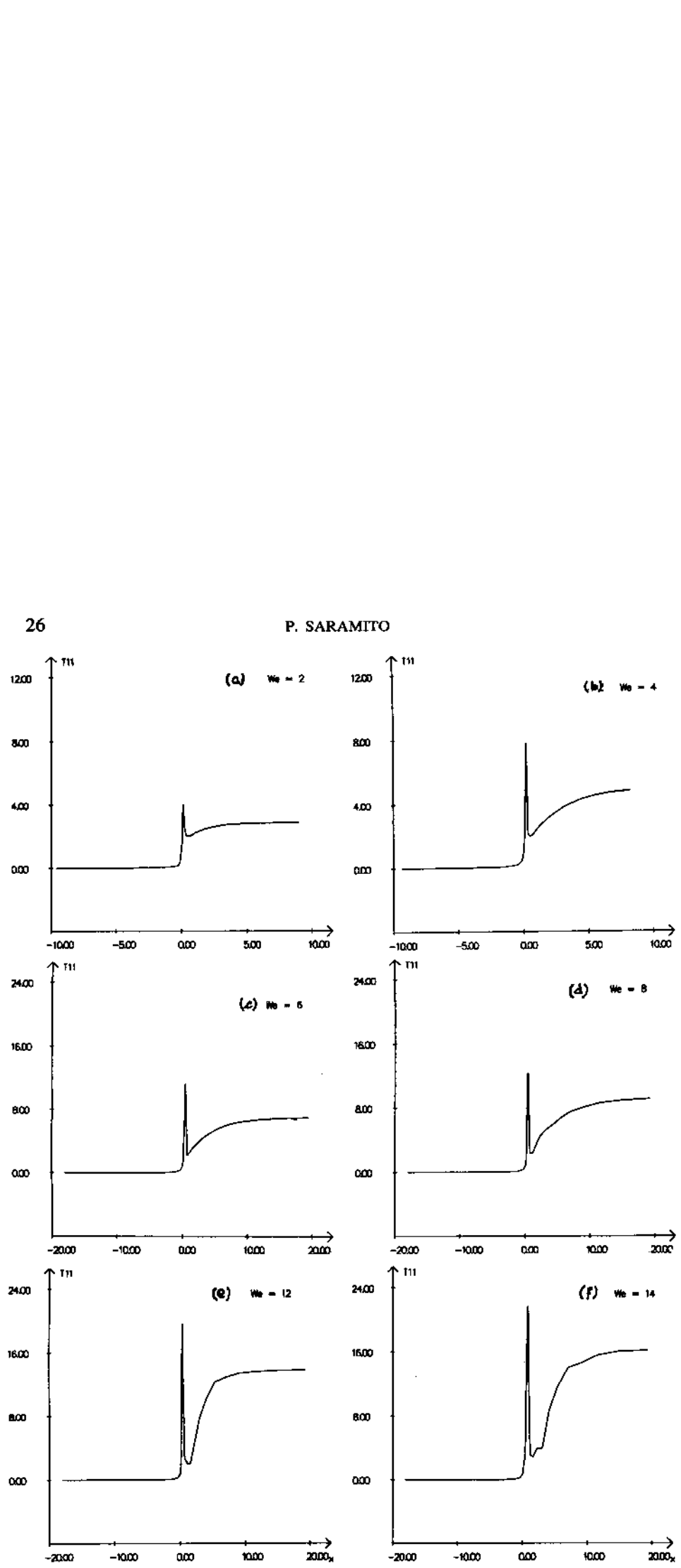


**Figure 6. — Stress component $\tau_{11}$ at the corner singularity.**

($F$) find $\psi \in H^1(\Omega)$, $\psi = \psi_\Gamma$ on $\Gamma$, such that :

$$(\mathbf{curl}\ \psi, \mathbf{curl}\ \xi) = (\mathbf{u}, \mathbf{curl}\ \xi), \quad \text{for all} \quad \xi \in H_0^1(\Omega) \tag{6.13}$$

where $\mathbf{u}$ denotes the velocity field, and $\psi_\Gamma$ the boundary condition given at the upstream and downstream section by :

$$\psi_\Gamma = \psi_0 \cdot \Phi\left(\frac{x_2}{r_i}\right), \quad 0 \leqslant x_2 \leqslant r_i, \quad i = 1, 2 \tag{6.14}$$

where

$$\Phi(x) = \begin{cases} 1/2(x-1)^2\,(x+2) & \text{for a plane domain} \\ (x-1)^2\,(x+1)^2 & \text{for an axisymmetric domain} \end{cases} \tag{6.15}$$

and

$$\psi_0 = \begin{cases} f_2\, r_2/3 & \text{for a plane domain} \\ f_2\, r_2^2/8 & \text{for an axisymmetric domain .} \end{cases} \tag{6.16}$$

On the axis, we may write :

$$\psi_\Gamma(x_1, 0) = \psi_0, \quad s_1 \leqslant x_1 \leqslant s_2 \tag{6.17}$$

and $\psi_\Gamma = 0$ on the wall.

The flow will be represented by the sketch of the computed streamlines, which are lines of $\psi/\psi_0$. The main flow in the central region corresponds to the case $\psi > 0$, since the recirculating zone is characterized by $\psi < 0$. The separating line between the two regions is determined by the condition $\psi = 0$.

The intensity of the recirculations is determined by the quantity

$$\min_{x \in \Omega} (\psi/\psi_0) .$$

When the velocity field is approximated by $\mathbf{u}_h \in V_h$, it is possible to approximate $\psi$, the unique solution of ($F$), by $\psi_h \in \Theta_h$.

Figure 7 shows the sketch of streamlines in a plane (7*a*) and an axisymmetric (7*b*) contractions, for the triangulation $T^{(3)}$. It can be seen in figure 8 that for the case of plane symmetry, the intensity and the length of the circulating zones appear to increase moderately when the Weissenberg number grows. This result confirms those given in the literature [4, 7].

Conversely, in the case of the axisymmetric contraction, it is well-known that viscoelastic liquids exhibit important vortex zones, which notably differ from the Newtonian behavior in such geometry. From our own results, we observe that, when the Weissenberg *We* increase, the center of the circulating zone moves from the re-entrant corner to the salient edge

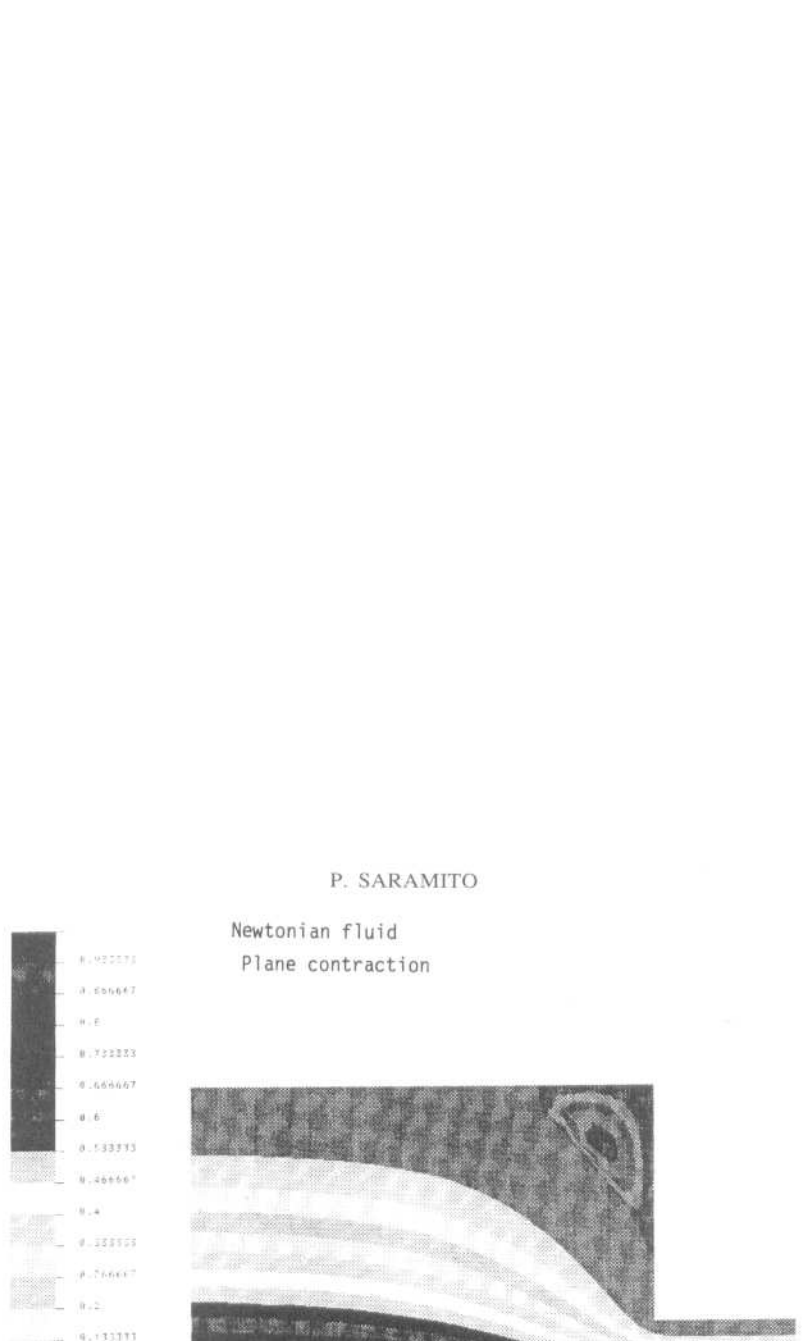


**Figure 7. — Streamlines for a slow Newtonian flow in a plane (a) and axisymmetric (b) abrupt contraction.**

**Figure 8. — Streamline in plane contraction ($We = 4$).**

($We = 20$, *fig.* 9) and the intensity of the recirculation is increasing. The focus zone then moves towards the upstream section and enlarges near the separating line, as can be seen in figure 10, at $We = 60$. It can finally be observed that the secondary flow zone leaves the re-entrant corner towards the salient edge with an axial increase of size in direction to the upstream section (*fig.* 11, $We = 90$).

Beyond the value of $We = 100$, the width $s_2 = 200$ does not permit the fully developed Poiseuille flow to be obtained at the limiting downstream section. Although no convergence problems were encountered for $We > 100$, the numerical experiments were not pursued beyond this value, mainly because of the necessity to extend the downstream flow zone. It should be outlined that, up to $We = 100$, stationary numerical solutions were still obtained by the algorithm.

### 6.4.3. *Velocity profiles on the axis*

A velocity overshoot for the first velocity component $u_1(x_1, 0)$, $s_1 \leq x_1 \leq s_2$ may be observed, in comparison with the downstream Poiseuille velocity profile. This overshoot phenomenon, which is not clearly apparent

Figure 9. — Streamline in axisymmetric contraction ($We = 10$).

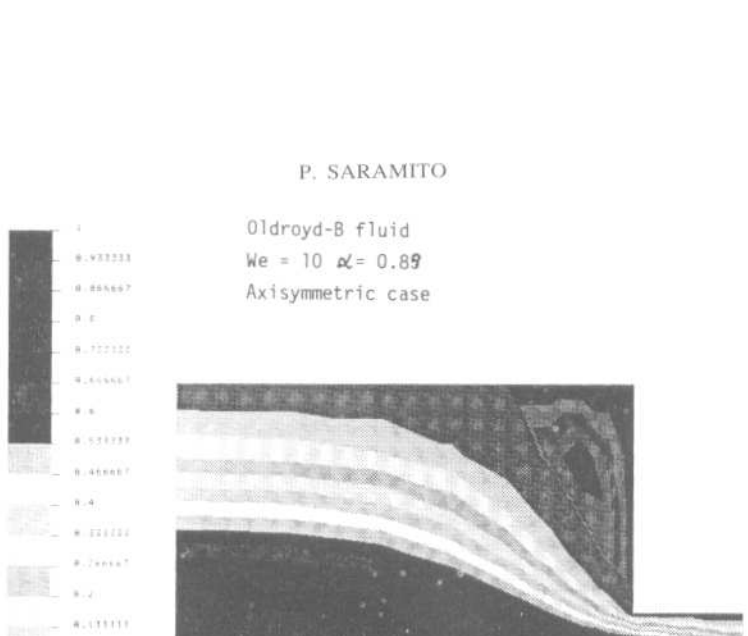


Figure 10. — Streamline in axisymmetric contraction ($We = 60$).

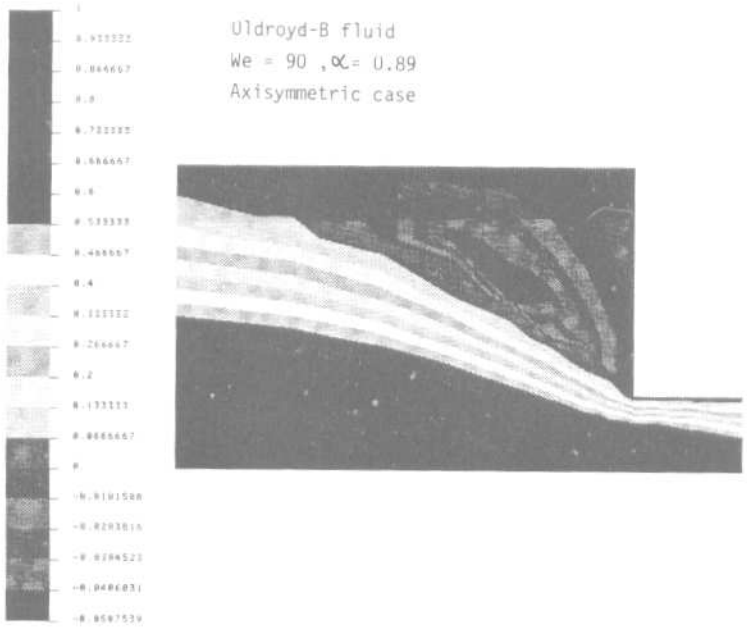


Figure 11. — Streamline in axisymmetric contraction ($We = 90$).

for a Newtonian fluid (*fig.* 12*a*) is found to increase fastly *versus* $We$. According to our results, the abscissa of the maximum value of $u_1$ moves towards the downstream section when the Weissenberg number increases (*figs.* 12*b* and 12*c*).

6.4.4. *Computing time*

On table 2, we have reported the CPU times for the converged solution at $We = 4$, as a function of the triangulation which has been used.

The low cost of our numerical procedure makes it practical for a workstation (e.g., Apollo DN 4000). The use of the C language allowed us to manage the storage area efficiently (by performing dynamic memory allocation and reclamation) : compacting procedures for the storage of the matrix resulting from the finite element method were used. The storage area cost was found to be linear in term of the size $N$.

## 7. CONCLUSION

In this paper, the major problems related to the numerical simulation of the flow of certain classes of constitutive equations have been considered. A

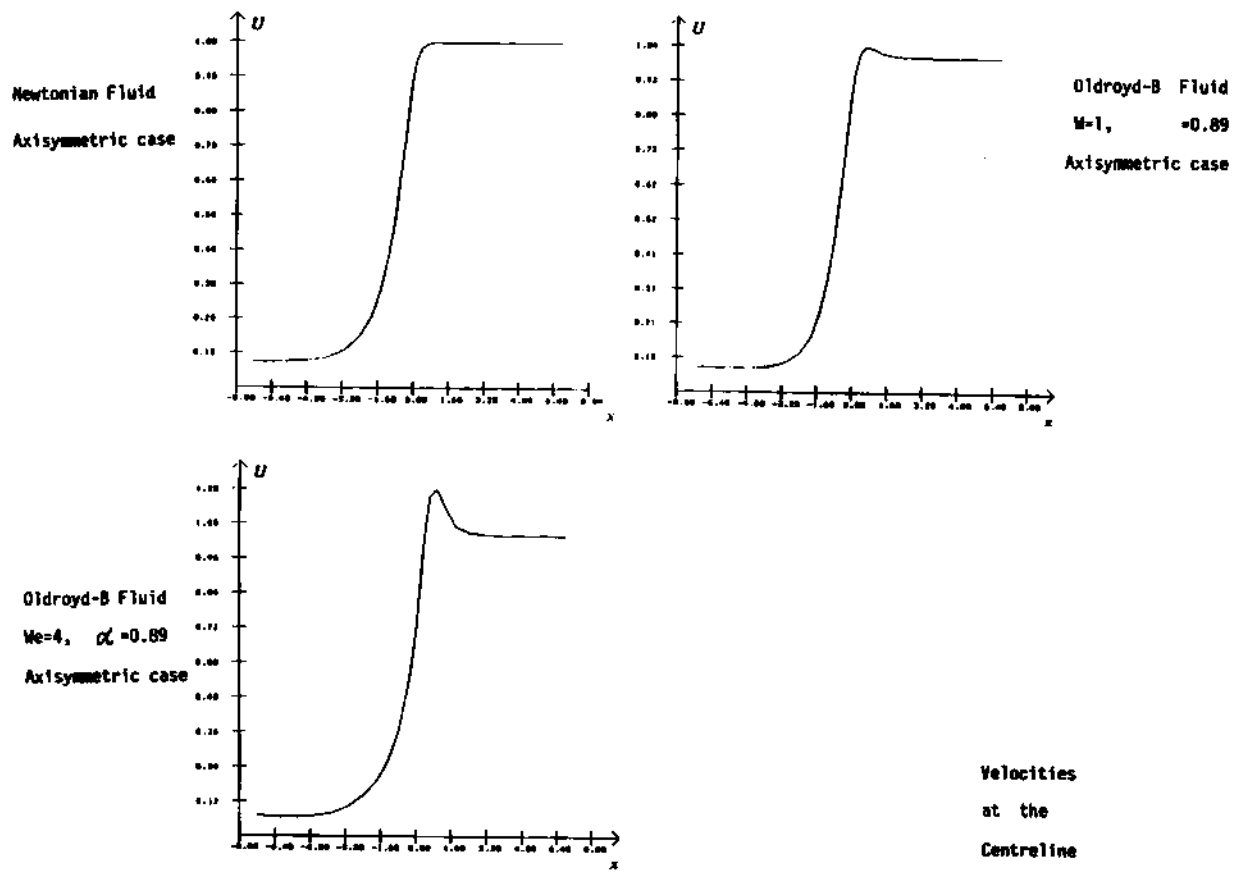


Figure 12. — Velocity profile on the axis.

Table 2. — Cost in computing time and storage area.

| Mesh | $N$ | CPU time |
|---|---|---|
| 0 | 945 | 55 s |
| 1 | 1 909 | 3 mm 28 s |
| 2 | 3 615 | 10 mm 16 s |
| 3 | 7 403 | 28 mm 32 s |

resolution procedure has been proposed, which enables to overcome the difficulties detailed at the beginning of the paper. The distinguishing features of the method may be summarized as follows :

i) The retained approximation for the unknowns verifies the compatibility conditions, leading to a well-posed problem when the Weissenberg number is zero. The elements chosen for the computation are robust : the incompressibility condition is verified exactly. Moreover, the particular element defined in section 4.2.2 is inexpensive, and may be interpreted in the context of finite difference schemes : this element is easy to use and could be inserted in existing codes.

ii) The approximate transport sub-problem related to the Oldroyd-B equations has been solved with appropriate schemes, giving stationary solutions at high Weissenberg numbers.

iii) Spectacular reduction of the CPU time has been obtained. The algorithm involving an ADI procedure appears to be robust and efficient : the computational steps are decoupled into standard sub-problems which can be solved in optimal conditions. Moreover, the space-time solution procedure enables us to consider in the next future time-dependent viscoelastic problems.

The method presented in this paper has been applied (and validated) to a difficult problem. Since the solution is singular due to the re-entrant corner of the contraction, we have had to generate meshes involving a high number of elements : our method made it practically tractable.

Our results have been found to be quantitatively consistent with those given by Marchal and Crochet [4]. Moreover, it has proved possible to compute on a workstation, for the first time to our knowledge, stationary solution of the flows of an Oldroyd-B fluid, at high Weissenberg numbers in the 4/1 axisymmetric contraction.

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